Erdős-Ko-Rado Theorems for Groups

Karen Meagher
(joint work with Bahman Ahmadi, Chris Godsil, Alison Purdy, Sarobidy Razafimahatratra, Peter Sin, Pablo Spiga and Pham Huu Tiep)

Department of Mathematics and Statistics
University of Regina

WinCom Virtual Colloquium, Mar. 25, 2021

Set systems

- A *k-set system* is a collection of subsets from $\{1, 2, \ldots, n\}$ in which each subset has size k.
- A k-set system is *intersecting* if for all sets A, B in the system

$$A \cap B \neq \emptyset.$$

An intersecting 3-set system from $\{1, \ldots, 6\}$:

123	124	125	126	134
135	136	234	235	236

In this system, every set has at least 2 elements from $\{1, 2, 3\}$.

Another intersecting 3-set system from $\{1, \ldots, 6\}$:

123	124	125	126	134
135	136	145	146	156

In this system, every set has the element 1.

Canonical Intersecting Set Systems

The easiest intersecting k-set system is the collection of all k-subsets that contain a fixed element.

$$\underline{1}\ \underbrace{\underline{*}\,\underline{*}\ldots\underline{*}}_{k-1 \text{ entries}}$$

This is called a *canoncial* intersecting k-set system and has size

$$\binom{n-1}{k-1}.$$

Is the canonical intersecting system the largest intersecting system?

Are there intersecting set systems of size $\binom{n-1}{k-1}$, other than the canonical set system?

The Answer

Theorem (Erdős-Ko-Rado Theorem)

Let $\mathcal{A}$ be an intersecting k-set system on an n-set. If $n > 2k$, then

1. $|\mathcal{A}| \leq \binom{n-1}{k-1}$,
2. *and $\mathcal{A}$ meets this bound if and only if it is canonically intersecting.*

Theorem (Erdős-Ko-Rado Theorem for t-intersecting)

Let $\mathcal{A}$ be a t-intersecting k-set system on an n-set. If n is large relative to t and k, then

1. $|\mathcal{A}| \leq \binom{n-t}{k-t}$,
2. *$\mathcal{A}$ meets this bound if and only if it is canonically t-intersecting.*

Canonical t-intersecting: $\underline{1}\ \underline{2} \ldots \underline{t}\ \underbrace{\underline{*}\ \underline{*} \ldots \underline{*}}_{k-t \text{ entries}}$

Kneser Graph

Define the Kneser graph $K(n, k)$

1. vertices are k-subsets of $\{1, \ldots, n\}$;
2. two k-sets are adjacent if they are disjoint.

An independent set/coclique in $K(n, k)$ is an intersecting set system.

What is the largest coclique in this graph?

What is the structure of a largest coclique in this graph?

Good Ol'Pete

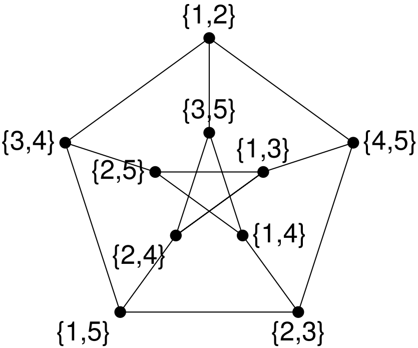


Figure: The Kneser Graph $K(5,2)$, or our old friend Petersen.

Properties of $K(n,k)$

1. There are $\binom{n}{k}$ vertices, each vertex has degree $\binom{n-k}{k}$.
2. It is vertex transitive and edge transitive.
3. A coclique (independent set) is an intersecting set.

Algebraic Graph theory

1. We can represent a graph X as a matrix $A(X)$.
2. Rows and columns are vertices (in the same order) and the
3. u, v entry of $A(X)$ is 1 if u and v are adjacent, and 0 otherwise.

Example for Kneser graph $K(5,2)$:

$$
\begin{array}{c}
 \\ 12 \\ 13 \\ 14 \\ 15 \\ 23 \\ 24 \\ 25 \\ 34 \\ 35 \\ 45
\end{array}
\begin{array}{c}
\begin{array}{cccccccccc} 12 & 13 & 14 & 15 & 23 & 24 & 25 & 34 & 35 & 45 \end{array} \\
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\end{array}
$$

Hoffman's bound/Delsarte's bound/Ratio bound

The **characteristic vector** of a set of vertices S in a graph is a 01-vector of length-$|V(G)|$; the v-entry is 1 if $v \in S$, and 0 otherwise. Denoted by v_S.

Ratio Bound

If X is a d-regular graph then

$$\alpha(X) \leq \frac{|V(X)|}{1 - \frac{d}{\tau}}$$

where τ is the least eigenvalue for the adjacency matrix for X.

If equality holds and S is a coclique of maximum size, then

$$v_S - \frac{|S|}{|V(X)|}\mathbf{1}$$

is a τ-eigenvector.

Ratio Bound for Kneser Graph

Bound on the size of a coclique in Kneser graph:

$$\alpha(K(n,k)) \leq \frac{\binom{n}{k}}{1 - \frac{\binom{n-k}{k}}{-\binom{n-k-1}{k-1}}} = \binom{n-1}{k-1}.$$

Characterization:

- Let v_i be the characteristic vector of the collection of all sets that contain i. The vectors $v_i - \frac{k}{n}\mathbf{1}$ are $-\binom{n-k-1}{k-1}$-eigenvectors.
- v_i span the $\binom{n-k}{k}$-eigenspace and the $-\binom{n-k-1}{k-1}$-eigenspace.
- The characteristic vector for any maximum coclique is a linear combination of the v_i.
- If $n > 2k$, the only linear combinations that give 01-vector with weight $\binom{n-1}{k-1}$ is v_i.

Can do this for t-intersection too, using a weighted adjacency matrix.

Intersecting Permutations

Two permutations $\sigma, \pi \in \mathrm{Sym}(n)$ *intersect* if for some $i \in \{1, \ldots, n\}$.

$$\sigma(i) = \pi(i) \quad \text{or} \quad \pi^{-1}\sigma(i) = i.$$

1. A permutation is a *derangement* if it fixes no points.
2. Permutations σ and π are intersecting if and only if $\pi^{-1}\sigma$ is **not** a derangement.

Intersecting Permutations

What is the size of the largest set of intersecting permutations?

What is the structure of the largest set of intersecting permutations?

The *canonical intersecting sets* are

$$S_{i,j} = \{\sigma \in \mathrm{Sym}(n) \,|\, i^\sigma = j\}.$$

1. If $i = j$, then $S_{i,i}$ is the stabilizer of i (this is a subgroup),
2. if $i \neq j$ it is a coset of a subgroup.
3. $S_{i,j}$ is an intersecting set of size $(n-1)!$.
4. Use $v_{i,j}$ for the characteristic vector of $S_{i,j}$.

A Simple Bound

Consider the following partition of the permutations in $\mathrm{Sym}(4)$:

(1)	(1,2,3,4)	(1,3)(2,4)	(1,4,3,2)
(1,2)	(1,3,4)	(1,4,2,3)	(2,4,3)
(1,3)	(1,4)(2,3)	(1,2)(3,4)	(2,4)
(1,4)	(2,3,4)	(1,3,2)	(1,2,4,3)
(2,3)	(1,2,4)	(1,3,4,2)	(1,4,3)
(3,4)	(1,2,3)	(1,3,2,4)	(1,4,2)

1. The first row is the sharply transitive group $C_4 = \langle(1,2,3,4)\rangle$.
2. No two permutations intersect in the first row.
3. The other rows are cosets of this subgroup; no two permutations intersect in a row.

Any maximum intersecting set of permutations will size at most $\frac{4!}{4} = 3!$.

Canonical Intersecting

Theorem (Deza and Frankl, 1977)

The size of the largest set of intersecting permutations is $(n-1)!$

Are there intersecting sets of permutation in $\mathrm{Sym}(n)$ with size $(n-1)!$ that are not canonical intersecting sets?

The Erdős-Ko-Rado Theorem for Permutations

Theorem

Let $\mathcal{P}$ be an intersecting set of permutations from $\mathrm{Sym}(n)$, then

1. $|\mathcal{P}| \leq (n-1)!$,
2. *and $\mathcal{P}$ meets this bound if and only if it is a canonically intersecting set of permutations.*

There are several proofs of this result.

- Cameron and Ku, 2003
- Larose and Malvenuto, 2004 (More general result)
- Wang and Zhang, 2007 (Clever use of clique-coclique bound)
- Godsil and Meagher, 2009 (Algebraic method, like Wilson's)

What about different groups?

A group has the *EKR-property* if the **size** of the largest set of intersecting permutations is the size of the largest stabilizer of a point.

A group has the *strict-EKR* property if the **only maximum** intersecting permutations are the stabilizer of a point or a coset of one.

❄ These properties depend on the action of the group.❄

Example

$\mathrm{Sym}(n)$ with its natural action on $\{1, \ldots, n\}$ has the strict-EKR property.

A non-EKR group

Example

The group $\mathrm{Sym}(8)$ acting on the ordered 4-sets from $\{1, \ldots, 8\}$ does not have the EKR property.
The set of all permutations that fix at least 5 of $[1..6]$ is intersecting and bigger.

- Subgroup that fixes the elements $\{1, 2, 3, 4\}$ has size $4! = 24$.

- The set that fixes at least 5 of $\{1, 2, 3, 4, 5, 6\}$

$$\underbrace{\binom{6}{6}2}_{\text{Fix all 6 elements.}} + \underbrace{\binom{6}{5}}_{\text{Pick 5 fixed elements.}} \underbrace{(2)}_{\text{Non-fixed to 7 or 8.}} \underbrace{(2)}_{\text{Place 7 and 8.}} = 26$$

Derangement Graph

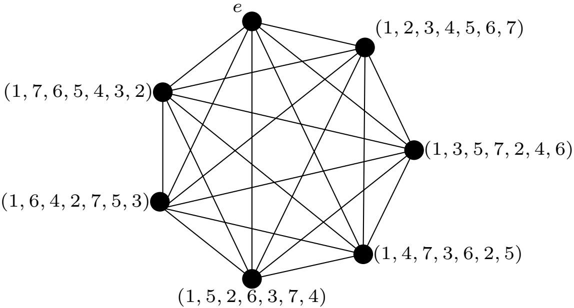
For any $G \leq \mathrm{Sym}(n)$ we can define a *Derangement Graph*.

- Γ_G denotes the derangement graph for a group G.
- The vertices are the elements of G.
- Vertices $\sigma, \pi \in G$ are adjacent if and only if $\pi^{-1}\sigma$ is a derangement. (So adjacent if **not** intersecting.)

The derangement graph depends on the action!

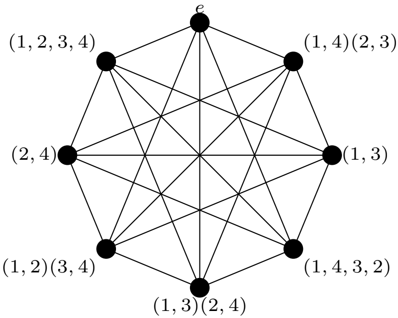
An intersecting set in G is a coclique in Γ_G.

Derangement Graph of $\mathbb{Z}_7$



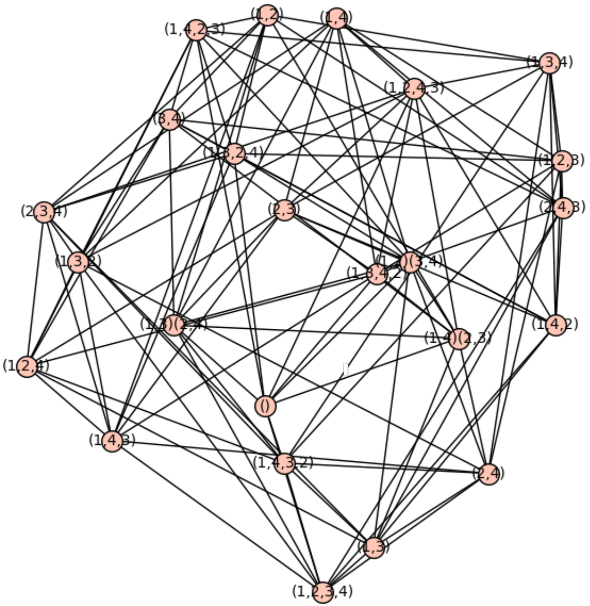
The graph $\Gamma_{\mathbb{Z}_7}$.

Derangement Graph of Dihedral Group



The graph $\Gamma_{D(4)}$.

Derangement Graph for $\mathrm{Sym}(4)$



Properties of the Derangement Graph

- Γ_G is vertex transitive.
- An intersecting set in G is a coclique in Γ_G.
- If G has a sharply 1-transitive set, then Γ_G has a clique of size n.
- Clique-coclique bound for a vertex transitive graph X

$$\alpha(X) \leq \frac{|V(X)|}{\omega(X)}.$$

If $G \leq \mathrm{Sym}(n)$ has a sharply 1-transitive subgroup, then $|S_{i,j}| = \frac{|G|}{n}$ and the canonical intersecting sets are maximum.

Cayley Graphs

*L*et G be a group and C a subset of G. Define the *Cayley Graph* $Cay(G,C)$ to be the graph with

- the vertices elements of G,
- and g,h are adjacent if gh^{-1} is in the set C.

1. A Cayley graph $Cay(G,C)$ is a *normal* Cayley graph if C is closed under conjugation.
2. If $der(G)$ is the set of derangements in G, then

$$\Gamma_G = Cay(G, der(G)).$$

   so Γ_G is a **normal** Cayley graph with connection set $Der(G)$ the set of derangements of G.
3. Γ_G is connected if and only if the derangements generate the group.
4. Γ_G is a union of graphs in that conjugacy class association scheme.

Eigenvalues of Cayley Graphs

Theorem

If $Cay(G, C)$ is a normal Cayley graph, then the eigenvalues of $Cay(G, C)$ are

$$\frac{1}{\chi(1)} \sum_{\sigma \in C} \chi(\sigma)$$

where χ is an irreducible character of G.

Example

Let $\mathbf{1}$ be the trivial character for G, then

$$\lambda_{\mathbf{1}} = \frac{1}{\mathbf{1}(1)} \sum_{g \in \mathrm{der}(G)} \mathbf{1}(g) = |\,\mathrm{der}(G)| = d.$$

This is the degree of the derangement graph.

Frobenius Groups

Example

If $G \leq \mathrm{Sym}(n)$ is a Frobenius group, then the spectrum of Γ_G is

$$\{n-1^{(k)}, \quad -1^{k(n-1)}\}.$$

The derangement graph of any Frobenius group is the union of k complete graphs on n vertices.

1. Any Frobenius group has the EKR property.
2. If $k > 2$ then it does not have the strict EKR property.
   There are n^k maximum cocliques, and n^2 canonical cocliques.

Frobenius

Example

Let $G = \mathrm{PGL}(2,q)$, the characters can be calculated:

Character	λ_1	λ_{-1}	ψ_1	ψ_{-1}	η_β	ν_γ
Eigenvalue	$\frac{q^2(q-1)}{2}$	$\frac{-q(q-1)}{2}$	$\frac{-q(q-1)}{2}$	$\frac{q-1}{2}$	q	0

1. $\mathrm{PGL}(2,q)$ has the EKR property.
   Use the ratio bound.
2. $\mathrm{PGL}(2,q)$ has the strict-EKR property.
   First, no coclique is in the ψ_1-eigenspace.
   Second, any coclique in the λ_1-eigenspace is a canonical coclique.

2-Transitive Subgroups

1. The permutation character is $\operatorname{fix}(g)$.
2. Define $\chi(g) = \operatorname{fix}(g) - 1$ (So χ = permutation $-$ trivial).
3. G is 2-transitive if and only if χ is irreducible.

$$\begin{aligned}\langle \chi, \chi \rangle &= \frac{1}{|G|} \sum_{g \in G} (|\operatorname{fix}(g)| - 1)^2 \\ &= \frac{1}{|G|} \sum_{g \in G} |\operatorname{fix}(g)|^2 - 2\frac{1}{|G|} \sum_{g \in G} |\operatorname{fix}(g)| + \frac{1}{|G|} \sum_{g \in G} 1 = 1\end{aligned}$$

4. The eigenvalue for χ is

$$\lambda_\chi = \frac{1}{\chi(1)} \sum_{g \in Der(G)} \chi(g) = \frac{-|Der(G)|}{n-1}$$

5. Define $E_\chi[g,h] = \chi(h^{-1}g)$; the *permutation module* is the span of the columns of E_χ and $E_{\mathbf{1}}$ (the all ones vector).

Apply the Ratio Bound

Theorem

Let G be a 2-transitive group acting on an n-set. If $\frac{-|Der(G)|}{n-1}$ is the least eigenvalue for Γ_G, then the largest intersecting set has size $\frac{|G|}{n}$

Proof. By the ratio bound

$$\alpha(\Gamma_G) \leq \frac{|G|}{1 - \frac{|Der(G)|}{-\frac{|Der(G)|}{n-1}}} = \frac{|G|}{n}.$$

Since G is transitive, then the stabilizer of a point has size $|G|/n$.

Theorem

Further, if only χ has eigenvalue $\frac{-|Der(G)|}{n-1}$, then the characteristic vector of any maximum coclique S is in the permutation module.

Overview of Method

Assume that G is a group with a 2-transitive action on a set of size n.

1. Prove the bound $\alpha(\Gamma_G) \leq \frac{|G|}{n}$ holds.
   - If G has a sharply transitive group, use clique-coclique bound.
   - Use ratio bound with $A(\Gamma_G)$,
   - Use ratio bound with a weighted adjacency matrix.
2. Show the characteristic vector for any maximum coclique is a linear combination of the vectors $v_{i,j}$.
   - The vectors $v_{i,j}$ span the ψ-module.
   - Show the characteristic vector for any maximum coclique is in this ψ-module.
3. Find all 01-vectors that are linear combinations $v_{i,j}$ with exactly $\frac{|G|}{n}$ ones.
   - Reduces to showing matrix has full rank.
   - or finding the kernel of this matrix.

3-types of EKR properties

A group G has the *EKR property* if the size of the maximum intersecting permutations is the size of the largest stabilizer of a point.

A group G has the *EKR-module property* if the characteristic vector of any maximum intersecting set is in the ψ-module.

A group G is *strict-EKR* if the only maximum intersecting permutations are the stabilizer of a point, or a coset of one.

EKR Property for 2-transitive group

Theorem (M, Spiga, Tiep)

All two transitive groups have the EKR property.

First we used the two reductions:

1. if a group has a sharply 1-transitive subgroup then it has the EKR property.
2. if G has a transitive subgroup H with the EKR property, then G has the EKR property.

A 2-transitive group G has a unique minimal normal subgroup ; either this subgroup is regular, or G is of almost simple type.

Table of 2-transtive groups of almost simple type

Line	Group S	Degree	Condition on G	Remarks
1	$\mathrm{Alt}(n)$	n	$\mathrm{Alt}(n) \leq G \leq \mathrm{Sym}(n)$	$n \geq 5$
2	$\mathrm{PSL}_n(q)$	$\frac{q^n-1}{q-1}$	$\mathrm{PSL}_n(q) \leq G \leq \mathrm{P\Gamma L}_n(q)$	$n \geq 2$, $(n,q) \neq (2,2),(2,3)$
3	$\mathrm{Sp}_{2n}(2)$	$2^{n-1}(2^n-1)$	$G = S$	$n \geq 3$
4	$\mathrm{Sp}_{2n}(2)$	$2^{n-1}(2^n+1)$	$G = S$	$n \geq 3$
5	$\mathrm{PSU}_3(q)$	q^3+1	$\mathrm{PSU}_3(q) \leq G \leq \mathrm{P\Gamma U}_3(q)$	$q \neq 2$
6	$\mathrm{Sz}(q)$	q^2+1	$\mathrm{Sz}(q) \leq G \leq \mathrm{Aut}(\mathrm{Sz}(q))$	$q = 2^{2m+1}$, $m > 0$
7	$\mathrm{Ree}(q)$	q^3+1	$\mathrm{Ree}(q) \leq G \leq \mathrm{Aut}(\mathrm{Ree}(q))$	$q = 3^{2m+1}$, $m > 0$
8	M_n	n	$M_n \leq G \leq \mathrm{Aut}(M_n)$	$n \in \{11,12,22,23,24\}$, M_n Mathieu group, $G = S$ or $n = 22$
9	M_{11}	12	$G = S$	
10	$\mathrm{PSL}_2(11)$	11	$G = S$	
11	$\mathrm{Alt}(7)$	15	$G = S$	
12	$\mathrm{PSL}_2(8)$	28	$G = \mathrm{P\Sigma L}_2(8)$	
13	HS	176	$G = S$	HS Higman-Sims group
14	Co_3	276	$G = S$	Co_3 third Conway group

Strict-EKR for 2-transitive groups

1. $\mathrm{Sym}(n)$ has strict EKR-property. (Cameron and Ku, Godsil and M.)
2. For $\mathrm{PGL}(n, q)$
   - for $n = 2$ has the strict-EKR property (M. and Spiga);
   - for $n \geq 3$ the maximum intersecting sets are either stabilizers of a point or a hyperplane (M. and Spiga, Spiga).
3. $\mathrm{PSL}(2, q)$ has the strict-EKR property (Long, Plaza, Sin, Xiang).
4. $\mathrm{Alt}(n)$ and the Mathieu groups have the strict EKR (Ahmadi, M.).
5. M_{11} on 12 points has strict EKR
6. $\mathrm{PSL}_2(11)$ on 11 and $\mathrm{Alt}(7)$ on 15 do not have strict EKR.

Fact

Not all two transitive groups have the strict-EKR property.

Every 2-transitive group has the EKR-module Property

A group has the **EKR-module property** if the characteristic vector of any maximum intersecting set in in the permutation module.

Theorem (M., Sin)

All 2-transitive groups have the EKR module property.

Corollary

For any 2-transitive group, the characteristic vector of any maximum intersecting set is a linear combination of the $v_{i,j}$.

Intersecting Subgroups

Let G be a 2-transitive group and S a maximum intersecting set.

1. S has the same inner distribution as the stabilizer of a point. (The inner distribution is the number of pairs of elements $g, h \in S$ with in gh^{-1} in a conjugacy class.)
2. If S is a group, then

$$\mathrm{ind}(\mathbf{1}_S)^G = \mathrm{ind}(\mathbf{1}_{G_x})^G$$

When does a group have non-conjugate subgroups that give the same induced representation?

For a 2-transitive group, are the maximum intersecting sets always subgroups or cosets of subgroups?

Intersection density

For a permutation group G (maybe make it transitive?) define the ratio of the size of a largest intersecting set to the size of a canonical intersecting set.

$$I(G) = \alpha(\Gamma_G)/|G_x|.$$

1. This ratio is 1 if and only if G has the EKR property.
2. If G is 2-transitive, then $I(G) = 1$.
3. If G acts on a set with prime order then $I(G) = 1$.
   (G has an element with order p, so a clique of size p.)

How big can this ratio be?

Recently Li, Song and Pantangi conjectured if $G \leq \mathrm{Sym}(n)$ is transitive then

$$\alpha(\Gamma_G)/|G_x| \leq \sqrt{n}.$$

They gave an example of a family of groups where this holds asymptotically.

Other Interesting Examples

Example (Razafimahatratra)

There is a group $G \leq \mathrm{Sym}(18)$ with $|G| = 324$ and $\Gamma_G = K_{108,108,108}$.

1. Maximum cocliques are 6 times larger than the stabilizer of a point.
2. This group has $\alpha(\Gamma_G)/|G_x| = 6$ (largest that we have found!).

There other are groups with

$$\Gamma_G = K_{\ell,\ell,\ldots,\ell}$$

1. the cocliques are much bigger than the stabilizer of a point.
2. G is imprimitive
3. These groups have the EKR-module property!

But, we only found 4 groups that have their derangement graph a complete tripartite graph. (Only 3 are counter examples to the conjecture.)

Other Results

Theorem (M., Razafimahatratra and Spiga)

A derangement graph for a transitive group $G \leq \mathrm{Sym}(n)$ with $n > 2$ is not bipartite.

Proof.

1. If Γ_G is bipartite, then its parts are H and xH where H is a normal subgroup G.
2. H has a normal covering number of 2, these have been mostly characterized.

Theorem (M., Razafimahatratra and Spiga)

A derangement graph for a transitive group must have a clique of size 3.

Questions I am thinking about

Is it true that for any 2-transitive group that any maximum intersecting set of permutations is either a group or a coset of group?

For 2-transitive groups, what are the boolean vectors in the χ-module? These can be considered to be the “Cameron-Leibler” sets for permutations.

Which 1-transitive groups have “interesting” intersecting set of permutations?

In a transitive group what is the largest set of permutations that is closed under taking conjugation?