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Maps, Δ-matroids and Rigidity Matroids for Graphs on Surfaces

Remi Cocou Avohou, **Brigitte Servatius, WPI**

Worcester Treegloo 2022

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1. Matroids vs. Δ-matroids

M is a Matroid

E a finite set – the *ground set* of M
$\mathcal{B} \subseteq \mathcal{P}(E)$ – the *bases* of M

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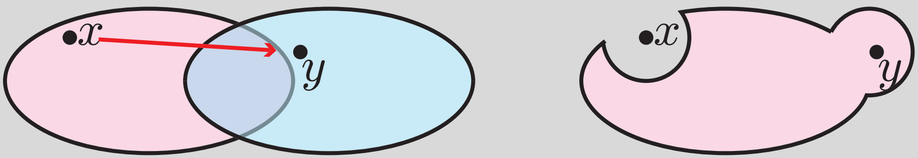
The *basis exchange axiom:*
$B_1, B_2 \in \mathcal{B}, x \in B_1 \setminus B_2 \Longrightarrow \exists y \in B_2 \setminus B_1$

$$(B_1 \cup \{y\}) \setminus \{x\} = B_1 \triangle \{x, y\} \in \mathcal{B}$$



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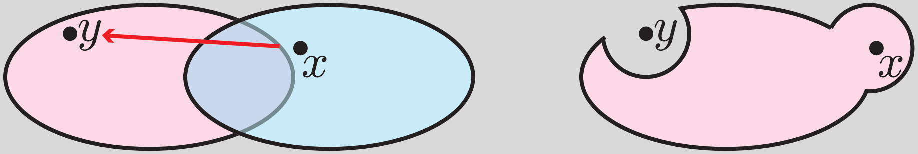

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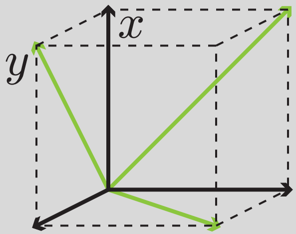

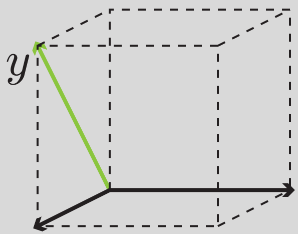

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$$(B_1 \cup \{y\}) \setminus \{x\} = B_1 \triangle \{x, y\} \in \mathcal{B}$$

Bases		–	$B \in \mathcal{B}$.
Independent sets	$\mathcal{I}$	–	$I \subseteq B \in \mathcal{B}$.
Dependent sets	$\mathcal{D}$	–	$D \notin \mathcal{I}$
Cycles (circuits)	$\mathcal{C}$	–	$C \in \mathcal{D}, C \not\subset D \in \mathcal{D}$
Spanning sets	$\mathcal{S}$	–	$S \supset B \in \mathcal{B}$.

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Whitney	1935	[20]	
W. T. Tutte	1971	[16]	(standard text)
D. J. A. Welsh	1976	[18]	(graph theory)
James Oxley	2011	[11]	(geometric/algebraic)
András Recski	1989	[13]	(applied approach)
Leonidas Pitsoulis	2014	[12]	

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D is a Δ-matroid

The *symmetric exchange axiom:*

$$F_1, F_2 \in \mathcal{F}, x \in F_1 \triangle F_2 \implies \exists y \in F_1 \triangle F_2$$

$$F_1 \triangle \{x, y\} \in \mathcal{F}$$

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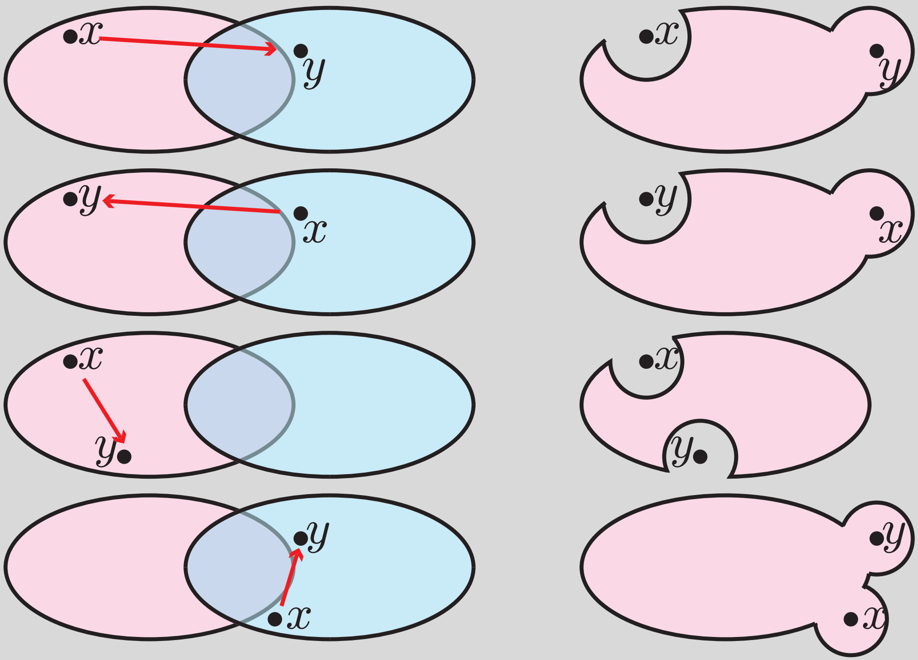

D is a Δ-matroid

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$$F_1, F_2 \in \mathcal{F}, x \in F_1 \triangle F_2 \implies \exists y \in F_1 \triangle F_2$$

$$F_1 \triangle \{x, y\} \in \mathcal{F}$$

Bouchet	1987	[3]	(Δ-matroids)
Bouchet	1998	[4, 5, 7, 6]	(multimatroids)
Dress & Havel	1986	[9]	(metroids)
Chandrasekaran	1988	[8]	(pseudometroids)

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2. Matroids in Δ-matroids

Given a Δ-matroid D, we find two matroids in D:

M_u, the *upper matroid*, whose bases are the feasible sets with largest cardinality

M_l, the *lower matroid*, whose bases are the feasible sets with least cardinality

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Theorem 1 *Let $M = (E, \mathcal{B})$ be a matroid with independent sets $\mathcal{I}$. Then $D = (E, \mathcal{I})$ is a Δ-matroid.*

The upper matroid is $(E, \mathcal{B})$ and the lower matroid $(E, \emptyset)$.

Theorem 2 *Let $M = (E, \mathcal{B})$ be a matroid with spanning sets $\mathcal{S}$. Then $D = (E, \mathcal{S})$ is a Δ-matroid.*

The upper matroid is $(E, \mathcal{P}(E))$ and the lower matroid $(E, \mathcal{B})$.

Theorem 3 *If $D = (E, \mathcal{F})$ is a Δ-matroid, $F \in \mathcal{F}$, then F is spanning in M_l and F is independent in M_u.*

Corollary 1 *If $M_u = (E, \mathcal{B}_u)$ and $M_l = (E, \mathcal{B}_l)$ are matroids, then for M_u and M_l to be upper and lower matroids of a Δ-matroid $D = (E, \mathcal{F})$ it is necessary that*
– every basis of M_u be spanning in M_l and
– every basis of M_l be independent in M_u.

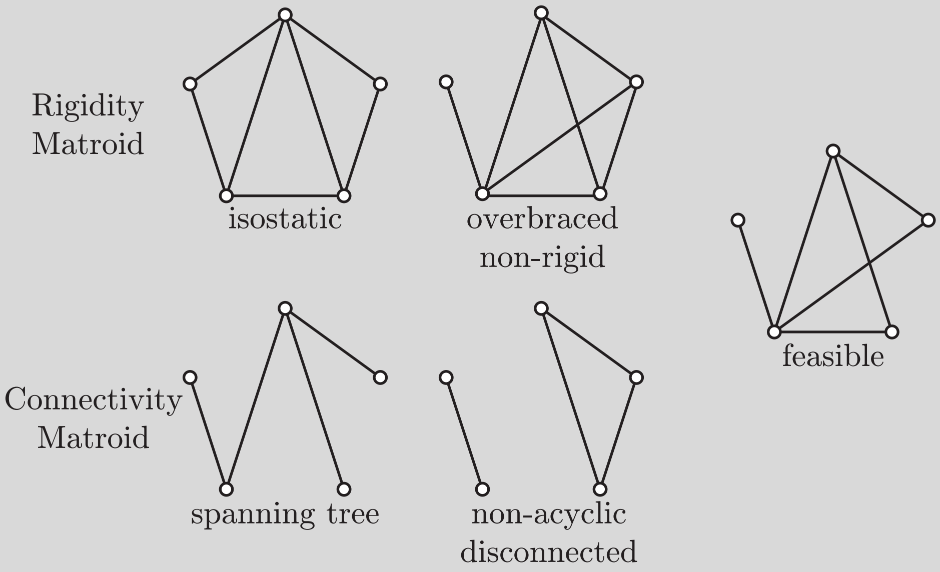
Upper and Lower matroids do not determine the Δ-matroid:

$$\{\{a,b\},\{a\},\{b\},\emptyset\} \qquad \{\{a,b\},\emptyset\}$$

3. Realization Problem

Given $M_l = (E, \mathcal{B}_l)$ and $M_u = (E, \mathcal{B}_u)$, construct $D = (E, \mathcal{F})$ realizing them.

An example with as many intermediate feasible sets as possible:



Theorem 4 $G = (V, E)$ *a connected simple graph.*
M_c *the connectivity matroid (cycle matroid)*
M_r *the 2-dimensional generic rigidity matroid*
$\mathcal{F}$: *F connected (spanning in M_c) not-overbraced (independent in M_r)*
Then $\mathcal{F}$ satisfies the symmetric exchange property.

Tool: A minimally overbraced graph is 2-connected.

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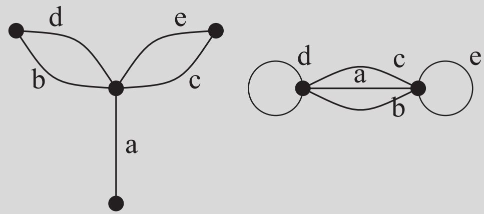
For this construction it is not necessary that a cycle in M_u be connected in M_l:

Example

$E = \{1, 2, 3, a, b, c\}$,
$M_u = U_{5,6}(E)$, $M_l = U_{2,3}(\{1, 2, 3\}) \oplus U_{2,3}(\{a, b, c\})$.
$D = (E, \mathcal{B}_u \cup \mathcal{B}_l)$ is a Δ-matroid.
M_u is a cycle.
M_l is disconnected.

A weaker condition: Every cycle in M_u is a union of cycles in M_l.

The weaker condition is necessary:



Two connectivity matroids on the same edge set.
M_u and M_l are matroids.
– Every basis of M_l is independent in M_u
– Every basis of M_u is spanning in M_l
But
Every cycle of M_u is not a union of cycles of M_l.

M_u and M_l are not the upper and lower matroids of any Δ-matroid.

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The weaker condition is necessary:

Theorem 5 *Let $D = (E, \mathcal{F})$, with upper matroid M_u and lower matroid M_l.*
Then every cycle in M_u is a union of cycles in M_l.

Theorem 6 *Given $M_u = (E, \mathcal{B}_u)$, $M_l = (E, \mathcal{B}_u)$, with*
Every cycle in M_u is a union of cycles in M_l.
Then every $B \in B_u$ is spanning M_l.
Then every $B \in B_l$ is independent in M_l.

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Necessary and Sufficient for Realization

Theorem 7 *Given* $M_u = (E, \mathcal{B}_u)$, $M_l = (E, \mathcal{B}_u)$.

M_u *and* M_l *realize the* Δ*-matroid* $D = (E, \mathcal{F})$
if and only if
Every cycle in M_u *is a union of cycles in* M_l.

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4. Quotients of Matroids

(Oxley [11]) $Q = (E, \mathcal{B}_Q)$ is a *quotient* of $M = (E, \mathcal{B}_M)$ if there is a matroid $N = (E \cup X, \mathcal{B}_N)$, $E \cap X = \emptyset$, with $M = N \setminus X$ and $Q = N/X$.

Theorem 8 *(Oxley) Q is a quotient of M if and only if every circuit of M is a union of circuits of Q.*

Corollary 2 *Given $M_u = (E, \mathcal{B}_u)$, $M_l = (E, \mathcal{B}_u)$.*

M_u and M_l realize the Δ-matroid $D = (E, \mathcal{F})$
if and only if
M_l is a quotient of M_u.

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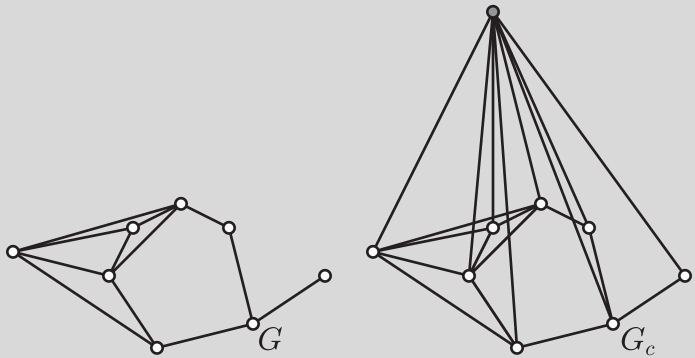
Corollary 3 [1] *The connectivity matroid of a graph is a quotient of the rigidity matroid.*



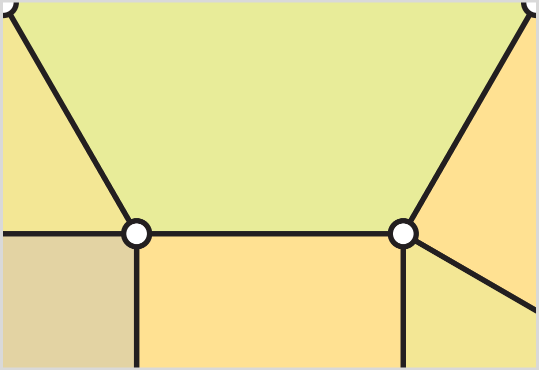
A graph G and its cone G_c.

Theorem 9 $M_r(G) = M_r(G_c) \setminus X$
$M_c(G) = M_r(G_c)/X$.

5. Combinatorial Maps and Δ-matroids

Tutte, in the introduction to his paper *What is a map?* [17] remarks

> Maps are usually presented as cellular dissections of topologically defined surfaces. But some combinatorialists, holding that maps are combinatorial in nature, have suggested purely combinatorial axioms for map theory, so that that branch of combinatorics can be developed without appealing to point-set topology.



Tutte's idea is that each edge of a map is associated with four flags, corresponding to the triangles in the barycentric subdivision.

The map can be uniquely described in terms of three perfect matchings. Two flags are matched if they differ in exactly one vertex.

Faces, Euler characteristic, and orientability can be treated combinatorially without appealing to topology.

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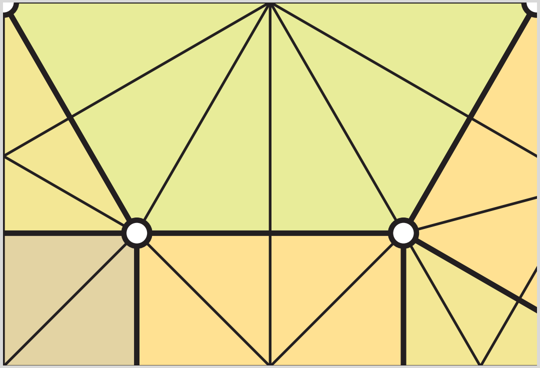

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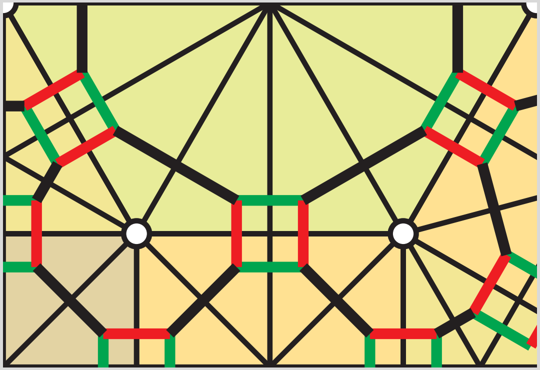

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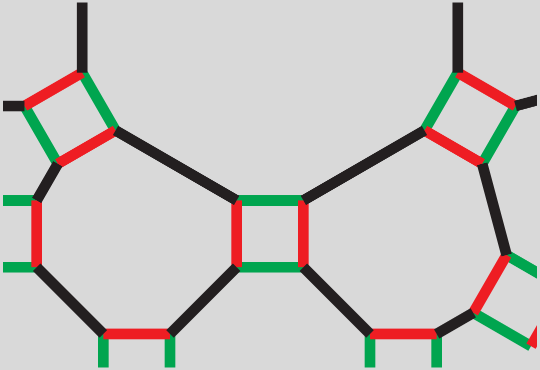

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Let Γ be a graph whose edges are partitioned into three classes R, G, and B which we color respectively red, green, and black. Γ is called *map graph* or a *combinatorial map* if the following conditions are satisfied:

1. Each color class is a perfect matching
2. $R \cup G$ is a union of 4-cycles
3. Γ is connected

The graph Γ is 3-regular and edge 2-connected. Γ may have parallel edges, although necessarily not red/green. Γ contains 2-regular subgraphs which use all the black edges of Γ, which we call *fully black* 2-regular subgraphs; $R \cup B$ and $G \cup B$ are examples, and there always exists a fully black Hamiltonian cycle.

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Proof of Hamiltonicity:

First note that a fully black 2-regular subgraph cannot contain any incident green and red edges, so every red/green quadrilateral intersects a fully black 2-regular subgraph in either two red, or two green edges.

Now consider a fully black 2-regular subgraph of Γ with the fewest connected components.

If there is not a single component, then there is a green/red quadrilateral which intersects the subgraph in, say, two red edges which belong to two different components, and swapping red and green on that quadrilateral reduces the number of components of the subgraph, violating minimality.

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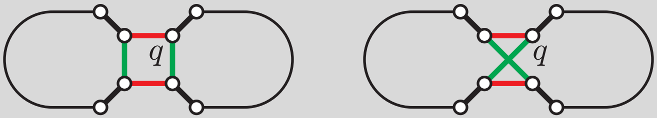
Theorem 10 [2] *Given a combinatorial map* $\Gamma(R, G, B)$*, let* E *be the set of quadrilaterals of* $R \cup G$*, and let* $\mathcal{F}$ *be the collection of subsets of* E *corresponding to the pairs of green edges in a fully black Hamilton cycle in* Γ*. Then* $(\mathcal{F}, E)$ *is a* Δ*-matroid.*

We have to show the symmetric exchange property. Let F_C and $F_{C'}$ be sets of quadrilaterals corresponding to fully black Hamiltonian cycles C and C'. Let $q \in F_C \triangle F'_C$, so the edges of quadrilateral q are differently colored in C and C', say red and green. There are two cases, either replacing in q the red edges in C with the green of C' results in two components or one. See Figure .



If it results in just one component, then take $q' = q$, and $F_c \triangle \{q, q'\} = F_c \triangle \{q\}$ is the set of red quadrilaterals of a fully black Hamiltonian cycle, and hence feasible, as required.

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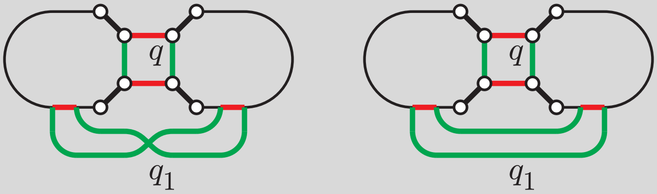
Otherwise, if there are two components, the Hamiltonian cycle of C' contains a non-black edge, say green, of a quadrilateral q_1, connecting those two components, and necessarily both red edges of q_1 are in C and both green edges of q_1 connect the components, and $q' \in C \triangle C'$.



Regardless of how the green edges of q_1 are placed, swapping the edges of both q and q' in C yields a new fully black Hamiltonian cycle, so the set $Q \triangle \{q, q_1\}$ is feasible, as required.

Since R, G and B are perfect matchings, the union of any two them induces a set of disjoint cycles. Let V be the set of cycles of $R \cup B$, E be the set of cycles of $R \cup G$, and V^* be the set of cycles of $G \cup B$. There is a graph (V, E) where incidence is defined between a red-black cycle and a red-green cycle if they share an edge, and, similarly, there is a graph (V^*, E) where incidence is defined between a green-black cycle and a red-green cycle if they share an edge. We say that Γ encodes the graph (V, E) and its geometric dual (V^*, E).

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Theorem 11 *Let $\Gamma(R, G, B)$ be a combinatorial map and let $D_\Gamma = (\mathfrak{F}, E)$ be its associated Δ-matroid.*
Then
the lower matroid of D_Γ is the cycle matroid of (V, E)
and
the upper matroid of D_Γ is the cocycle matroid of (V^*, E).

PROOF: Given $\Gamma(R, G, B)$, recall that the feasible sets of D consist of RG quadrilaterals whose R edges are contained in a fully black Hamilton cycle of Γ.

Any fully black Hamilton cycle C of Γ must contain the red edges corresponding to a spanning tree of (V, E) as well as the green edges corresponding to a spanning tree of (V^*, E). So the minimal number of red edges in C is $2(|V| - 1)$, while the maximal number is $2(|E| - |V^*| + 1)$.

The edge sets of the spanning trees of (V, E) are the bases of its cycle matroid, while the complements of edge sets of spanning trees in (V^*, E) are the bases of the cocycle matroid of (V^*, E). $\square$

• The difference in rank of the upper and lower matroid of $(\mathfrak{F}, E)$ is given by

$$(|E| - |V^*| + 1) - (|V| - 1) = 2 - \chi,$$

– χ is the Euler characteristic.

•If Γ is bipartite, all feasible sets of $D_\Gamma = (\mathfrak{F}, E)$ must have the same parity –
since exchanging a red and green pair of edges always disconnects a Hamilton cycle of a bipartite Γ.

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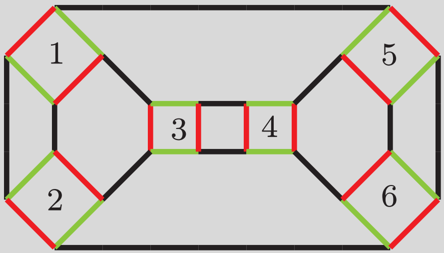
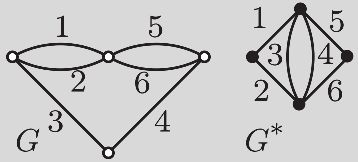
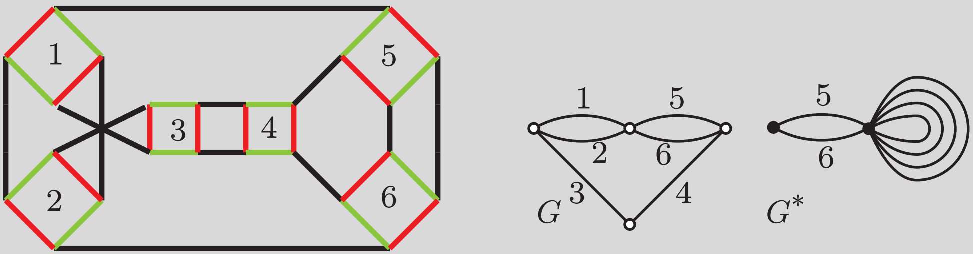




$$\mathfrak{F} = \{\{1,3,4\},\{1,3,5\},\{1,3,6\},\{1,4,5\},\{1,4,6\},$$
$$\{2,3,4\},\{2,3,5\},\{2,3,6\},\{2,4,5\},\{2,4,6\},\{3,4,5\},\{3,4,6\}\}$$
.

- $\mathfrak{F}$ is the set of spanning trees of G and at the same time the set of co-trees of G^* so
  – all feasible sets have the same size and
  – the upper and lower matroid are identical.



$$\mathfrak{F} = \{\{1,3,4\},\{1,3,5\},\{1,3,6\},\{1,4,5\},\{1,4,6\},$$
$$\{2,3,4\},\{2,3,5\},\{2,3,6\},\{2,4,5\},\{2,4,6\},\{3,4,5\},\{3,4,6\},$$
$$\{1,2,3,4,5\},\{1,2,3,4,6\}\}$$

The lower matroid is again the cycle matroid of G,
but the upper matroid is the co-cycle matroid of G^*,
The upper matroid has rank 5 and contains exactly one cycle, namely $\{5,6\}$,
which is a minimal cutset of G^* and also a cycle in G.

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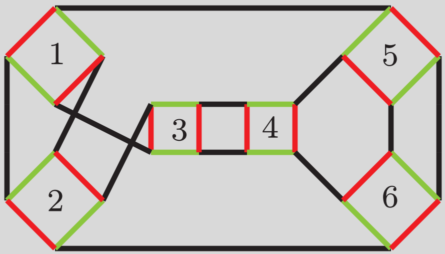

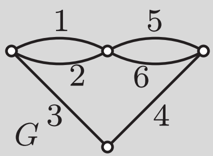

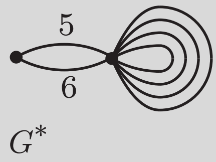

The Δ-matroid associated to this the map has,
in addition to the feasible sets of the previous example,
the feasible set $\{1, 2, 3, 4\}$, whose parity is even, while the parity of all other feasible sets is odd,
so this map is not orientable.

As is clear from these examples, the map cannot, in general be recovered from the Δ-matroid information, since the upper or lower matroid do not even determine the graph. Non-isomorphic graphs may have identical cycle-and co-cycle matroids. It is easy to check that $\mathfrak{F}$ is also a list of spanning trees for the graph G', but G is not isomorphic to G'.

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However, if both G and G^* are 3-connected, then the map is uniquely recoverable from the Δ-matroid information.

Theorem 12 *Let D be the Δ-matroid of a map M with 2-connected upper- and lower matroid. Then M is determined by D.*

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PROOF: By Whitney's theorem [19], upper and lower matroid uniquely determine G and G^*. To recover M from D, we need to specify a rotation system for each vertex v of G.
To determine if two edges e and f with endpoint v follow each other in the rotation about v, it is enough to check if e and f are both incident in G^*, since the vertex co-cycles of G^* correspond to the facial cycles of the embedded G.
Now re-construct the map graph $\square$

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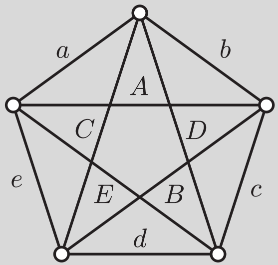
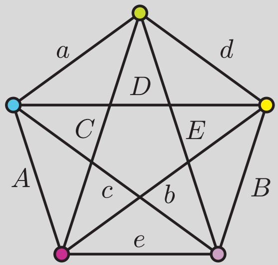




a A	b D	c B	d E	e C
c B	d E	e C	a A	b D
e C	a A	b D	c B	d E
b D	c B	d E	e C	a A
d E	e C	a A	b D	c B

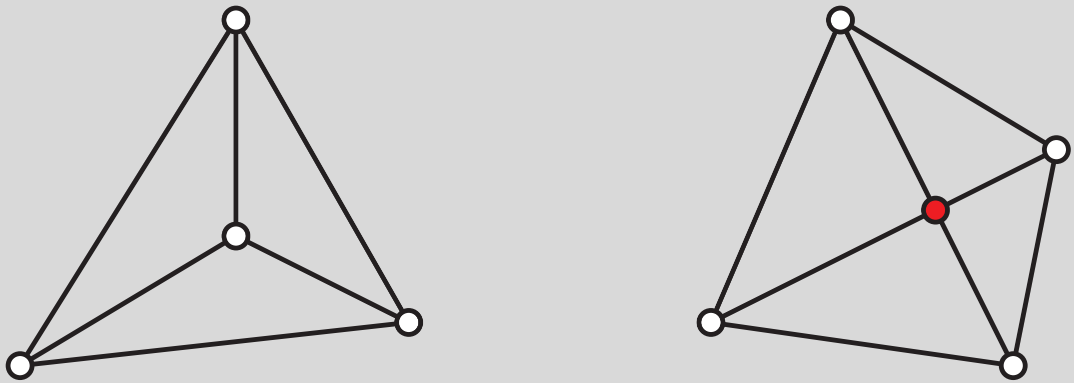
For example the lower matroid could be the cycle matroid of K_5, while the upper matroid is the co-cycle matroid of K_5 as well, so this matroid information gives us the graphs G and G^* depicted in Figures . By the method in the proof of Theorem 12 the map M is easily recovered.

6. Rigidity matroid for graphs on surfaces

Represent the surface by a polygon P with boundary identifications
Given $G(V, E)$, embed V and represent E by straight line segments on P.
Straight line segments might cross.

Miracle in the Plane

Planar rigidity cycles dualize into rigidity cycles.



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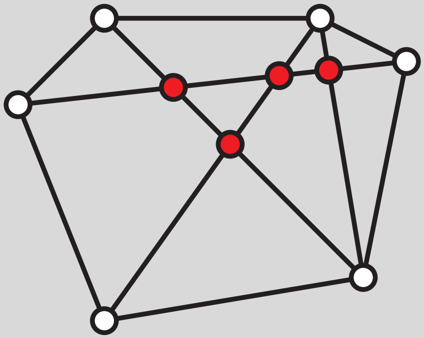

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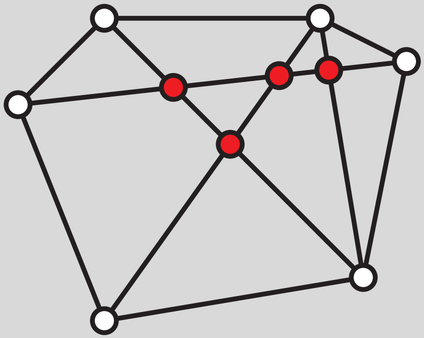
Rigidity matroid for graphs whose vertices are embedded in the plane

The rigidity matroid is the Dilworth truncation of two cycle matroids of G. [14, 15]
Equivalently, the rigidity matroid is the Dilworth truncation of M_l and M_u.



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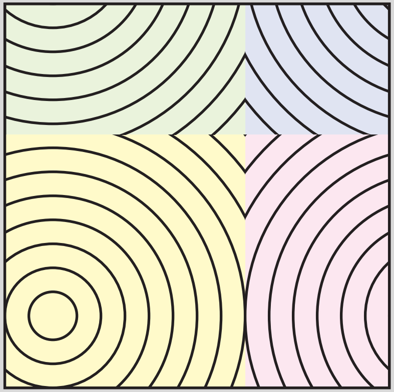
What is the probability that two points in the unit square torus have shortest distance intersecting the boundary square.



So the 4D volume of the collection of points

$$(x_0, y_0, x_1, y_1) \subseteq [0,1]^4$$

with this property is

$$\begin{aligned} V &= 4\int_0^{1/2}\int_0^{1/2}((2x+1)/2)((2y+1)/2)\ dx\ dy \\ &= \left[\int_0^{1/2} 2x+1\ dx\right]\left[\int_0^{1/2} 2y+1\ dy\right] \\ &= [1/4+1/2]^2 = 9/16 \end{aligned}$$

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Rigidity matroid for graphs whose vertices are embedded on a compact surface

Represent the surface by a polygon P with boundary identifications
Given $G(V, E)$, embed V and represent E by straight line segments on P.
Straight line segments might cross.
A.a.s. this yields a map M. The rigidity matroid for G on P is (a truncation) of the union of M_u and M_l.

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Handle slides

[10]
Let $D = (E, \mathcal{F})$ be a set system, and $a, b \in E$ with $a \neq b$. We define D_{ab} to be the set system $(E, \mathcal{F}_{ab})$ where

$$\mathcal{F}_{ab} = \mathcal{F} \Delta \{X \cup a | X \cup b \in \mathcal{F} \text{ and } X \subseteq E - \{a, b\}\}.$$

We call the move taking D to D_{ab}
a handle slide taking a over b.

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subsection*Normal form

For each binary Δ-matroid D, there is a sequence of handle slides taking D to some $D_{i,j,k,l}$ where i is the size of the ground set minus the size of a largest feasible set, l is the size of a smallest feasible set, $2j+k$ is difference in the sizes of a largest and a smallest feasible set. Moreover, $k=0$ if and only if D is even, and if D is odd then every value of j from 0 to $\lfloor \frac{w}{2} \rfloor$, where w is the difference between the sizes of a largest and a smallest feasible set, can be attained.

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